Transverse stability issues in Hamiltonian partial differential equations

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The famous Korteweg- de Vries (KdV) equation

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0$$

is a one dimensional asymptotic model obtained from the (much more complicated) water-waves system. The KdV equation has a well-known particular solution

$$S_c(t,x) = cQ(\sqrt{c}(x-ct)), \quad c>0, \quad Q(x) = 3\mathrm{ch}^{-2}(x/2) \tag{1}$$

called a solitary wave (c represents the propagation speed). In the remarkable work by Amick-Kirchgässner [1], it is shown that the full water-waves system still has one dimensional solitary waves of type (1).

The orbital stability of the KdV solitary wave $S_c(t,x)$ was obtained by Benjamin [2]. When studying the stability of the KdV solitary wave under transverse perturbations, Kadomtsev and Petviashvily introduced in [3] the two dimensional models

$$\partial_x(\partial_t u + u\partial_x u + \partial_x^3 u) \pm \partial_y^2 u = 0 \tag{2}$$

called KP-I and KP-II equations depending on the sign in front of $\partial_y^2 u$ (the sign plus gives KP-II while the sign minus KP-I). Clearly the KdV solitary wave $S_c(t,x)$ solves (2) as well. The formal analysis in [3] leads to the believe that the KdV solitary wave is stable as a solution of the KP-II equation and unstable as a solution of the KP-I equation.

In [4] Zakharov obtained an explicit solution of the KP-I equation demonstrating a certain form of the instability of the KdV solitary wave (1) as a solution of the KP-I equation. In this contribution we will present a different approach to the Zakharov result which has the great advantage to generalize to proving the transverse instability of the Amick-Kirchgässner solitary wave as a solution of the full two dimensional water-waves system. We shall also explain that for small speeds and fixed period transverse perturbations the KdV soliton is in fact stable as a solution of the KP-I equation. We shall also discuss the stability of the KdV solitary wave as a solution of the KP-II equation.

References

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