# Torsion homology growth in arithmetic groups

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To quote Jordan Ellenberg [4]: "I was raised to think of torsion classes in homology as a terrifying mystery that one dealt with by tensoring with the rational numbers as quickly as possible. But our knowledge about these things is actually starting to accumulate!"

Various recent works, see e.g. [2, 5, 8, 1, 6, 7], indeed show that certain arithmetic groups can have 'a lot' of torsion in their homology. Among these groups are the finite index (congruence) subgroups of $SL_3(\mathbb{Z})$ or $SL_2(\mathbb{Z}[i])$.

In particular, for $\Gamma_0(N) = \left\{ \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in PSL_2(\mathbb{Z}[i]) \mid N|c \right\}$ ($N \in \mathbb{Z}[i]$), homology reduces to the abelianization $\Gamma_0(N)^{ab} = \Gamma_0(N)/[\Gamma_0(N) : \Gamma_0(N)]$. This is a finitely generated $\mathbb{Z}$-module with torsion part $\mathrm{Tors}(\Gamma_0(N)^{ab})$. Akshay Venkatesh and I [2] have conjectured that, as $N$ tends to $\infty$ among primes, we have:

$$\frac{\log \#\mathrm{Tors}(\Gamma_0(N)^{ab})}{|N|^2} \to \frac{\lambda}{8\pi}, \quad \lambda = L(2, \chi_{\mathbb{Q}(i)}) = 1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \dots$$

More generally one may ask: *when does the amount of torsion in the homology of an arithmetic group grow exponentially with the covolume ?* We propose conjecturally precise conditions. This contribution presents ideas for how to attack this conjecture and discusses recent progress towards it.

This interacts with more classical questions of geometry (analytic torsion, Gromov-Thurston norm, (higher) cost, rank and deficiency gradient ...) and number theory (BSD conjecture, ABC conjecture ...). A big motivation is provided by (one of) Peter Scholze's recent breakthrough(s) [9]: *very roughly* a mod $p$ torsion class in $\mathrm{Tors}(\Gamma_0(N)^{ab})$ parametrizes a field extension $K/\mathbb{Q}(i)$ whose Galois group is $GL_2(\mathbb{F}_p)$. Moreover, it is anticipated that there is a corresponding 'torsion Langlands program', see [3].

## References

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