# The Arithmetic and Topology of Differential Equations

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The talk will describe arithmetic properties of solutions of linear differential equations, especially Picard-Fuchs equations, and their connections with modular forms, mirror symmetry, and invariants of algebraic varieties. (No prior knowledge of any of these topics will be assumed.) There will be cross-connections to many of the great achievements of Friedrich Hirzebruch, including his proportionality principle, his Riemann-Roch theorem, his resolution of the cusps of Hilbert modular surfaces, and his work on hypergeometric differential equations.

Our starting point is Apéry's astounding discovery that the solutions of the recursion

$$(n+1)^3\, A_{n+1} \;=\; (34n^3+51n^2+27n+5)\, A_n \;-\; n^3\, A_{n-1} \qquad (n\geq 0),$$

with initial value $A_0 = 1$, are all integers. He proved this from the obviously integral explicit formula $A_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{n}^2$, but gave only a rather obscure proof of the latter. (See [2] for the amusing full story.) Soon afterwards Beukers gave a much more enlightening proof by showing that Apéry's recursion had a *modular* solution: there exist a modular function $t(\tau)$ and a modular form $f(\tau)$ such that $f(\tau) = \sum_{n=0}^{\infty} A_n t(\tau)^n$, and the integrality of the $A_n$ follows because both $t$ and $f$ have integral expansions as power series in $q = e^{2\pi i\tau}$. (The way this works will be explained in the talk; see also [1], 61–66.) This divisibility miracle for linear recursions is extremely rare: in [3] I looked at the first 100 million cases of a certain 3-parameter family of recursions similar to Apéry's that had been discovered by Beukers and the integrality occurred for only 7 of them, all of which had modular parametrizations. Thus one could think (and for a long time I did) that the integrality phenomenon is equivalent to modularity. But the situation is more subtle. For instance there is a recursion of the same form as Apéry's and Beukers's (though now involving polynomials with coefficients in $\mathbb{Q}(\sqrt{17})$ rather than $\mathbb{Q}$) due to Irene Bouw and Martin Möller that they proved in an indirect way has an integral solution, but here Beukers's modularity argument does not work: there *is* a modular parametrization, but it involves a non-arithmetic modular group for which $t$ and $f$ do *not* have integral, or even algebraic, $q$-expansions. The solution to this conundrum, found in joint work with Möller, turned out to be that one had to embed the underlying modular curve into the Hilbert modular surface for $\mathbb{Q}(\sqrt{17})$ and use Hirzebruch's resolution of the cusp singularities! Yet other cases, like those associated to the families of Calabi-Yau 3-folds studied in mirror symmetry, remain mysterious. Some other connections with mirror symmetry will also be discussed, including the proof (in joint work with Vasily Golyshev) of many cases of the "Gamma Conjecture," which predicts that certain real numbers coming from the asymptotics of solutions of linear differential equations are given by a specific genus (in the sense of Hirzebruch) of the associated mirror manifolds. In yet another direction, again from joint work with Golyshev, I will explain that the Apéry sequence can be interpolated in a natural way and that its value at $n = 1/2$ is then related to a special value of the $L$-series of a modular form, and several further arithmetic surprises will also be described if time permits.

## References


[1] J. Bruinier, G. Harder, G. van der Geer, D. Zagier. *The 1–2–3 of Modular Forms: Lectures at a Summer School in Nordfjordeid, Norway*, (ed. K. Ranestad), Universitext, Springer-Verlag, Berlin–Heidelberg–New York (2008), 280 pages.

[2] A. van der Poorten. A proof that Euler missed... Apéry's proof of the irrationality of $\zeta(3)$. An informal report. Mathematical Intelligencer 1 (1979), 195–203.

[3] D.B. Zagier. Integral solutions of Apéry-like recurrence equations. In *Groups and Symmetries: From the Neolithic Scots to John McKay*. CRM Proceedings and Lecture Notes 47 (2009), Amer. Math. Society, 349–366.