Flexible polyhedra and their volumes

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Consider a closed polygonal line in plane. Regard the edges of this line as rigid bars and the vertices of this line as hinges that allow the angles between consecutive bars to change arbitrarily. Deformations of such mechanism that are not induced by ambient rotations of the plane are called *flexions.* It is well known that any triangle is rigid, i. e., admits no flexions, and a generic polygon with at least four edges is flexible.Now, consider a closed polyhedral surface in three-space, regard the faces of this surface as rigid plates and the edges of this surface as hinges that allow the dihedral angles between adjacent plates to change arbitrarily. Such mechanism is called a *flexible polyhedron* if it admits a deformation not induced by an ambient rotation of three-space. It turns out that the situation changes drastically when we pass from dimension 2 to dimension 3. For instance, a generic polyhedron of any combinatorial type is rigid. Moreover, it is rather hard to construct flexible polyhedra. The first examples of such sort were flexible self-intersecting polyhedral surfaces with the combinatorial type of an octahedron constructed by Bricard [1]. The first example of an embedded flexible polyhedral surface was constructed by Connelly [2]. The concept of a flexible polyhedron can be easily generalized to higher dimensions. Nevertheless, until recent results by the speaker only three- and four-dimensional examples were known. One of the most amazing conjectures concerning flexible polyhedra is the so-called *Bellows conjecture* claiming that the volume of any flexible polyhedron is constant during the flexion. The proof of this conjecture for flexible polyhedra in the Euclidean three-space by Sabitov [8], [9] was one of the most important breakthroughs in theory of flexible polyhedra; another proof was obtained in [3]. Notice that the area of a flexible polygon in plane is by no means constant, i. e., the analog of the Bellows conjecture in dimension 2 is obviously false.In this talk we shall describe the overall situation in theory of flexible polyhedra. In particular, we shall discuss the progress in several problems in this area achieved in a series of recent papers by the speaker [4]–[7]. This will include:

- The first examples of self-intersecting flexible polyhedra in dimensions 5 and higher in all spaces of constant curvature, i. e., in the Euclidean spaces $\mathbb{E}^n$, in the Lobachevsky spaces Λ^n, and in the open hemispheres $\mathbb{S}^n_+$.
- The first examples of embedded flexible polyhedra in the open hemispheres $\mathbb{S}^n_+$ for all $n \geq 4$. The volumes of these polyhedra change during the flexions, which disproves the Bellows conjecture in $\mathbb{S}^n_+$.
- The proof of the Bellows conjecture for flexible polyhedra in Euclidean spaces of dimensions 4 and higher, and for bounded flexible polyhedra in odd-dimensional Lobachevsky spaces.

The proofs of these results establish connections of theory of flexible polyhedra with various branches of modern mathematics, in particular, with algebraic geometry, combinatorial topology, elliptic functions, and complex analysis.

References

[1] R. Bricard. Mémoire sur la théorie de l'octaèdre articulé. J. Math. Pures Appl. 5:3 (1897), 113–148.

[2] R. Connelly. A counterexample to the rigidity conjecture for polyhedra. Inst. Hautes Études Sci. Publ. Math. 47 (1977), 333–338.

[3] R. Connelly, I. Sabitov, A. Walz. The Bellows Conjecture. Beitr. Algebra Geom. 38:1 (1997), 1–10.

[4] A.A. Gaifullin. Generalization of Sabitov's theorem to polyhedra of arbitrary dimensions. Discrete Comput. Geom. 52:2 (2014), 195–220, arXiv: 1210.5408.

[5] A.A. Gaifullin, Flexible cross-polytopes in spaces of constant curvature. Proc. Steklov Inst. Math. 286 (2014), 77–113, arXiv: 1312.7608.

[6] A.A. Gaifullin. Embedded flexible spherical cross-polytopes with non-constant volumes. Proc. Steklov Inst. Math. 288 (2015), 56–80, arXiv: 1501.06198.

[7] A.A. Gaifullin. The analytic continuation of volume and the Bellows conjecture in Lobachevsky spaces. Sb. Math. 206:11 (2015), to appear, arXiv: 1504.02977.

[8] I.Kh. Sabitov. Volume of a polyhedron as a function of its metric. Fund. and Applied Math. 2:4 (1996), 1235–1246 (in Russian).

[9] I.Kh. Sabitov. A generalized Heron–Tartaglia formula and some of its consequences. Sb. Math. 189:10 (1998), 1533–1561.