Complex Brunn-Minkowski theory and its applications in geometry.

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The classical Brunn-Minkowski theorem is an inequality for volumes of convex sets. Its original formulation is in terms of the Minkowski sum of convex sets. It can also be formulated as the statement that the volumes of the vertical slices of a convex body in Euclidean space form a log-concave function of the base point of the slice. This way the theorem does not presuppose any notion of addition, or other group operation, but starts from a notion of convexity. This makes it natural to investigate analogous statements for other notions of convexity, like holomorphic convexity or pseudoconvextity that arise in complex analysis. As it turns out, the corresponding inequalities no longer concern (primarily) volumes of sets or manifolds, but rather L^2-norms of holomorphic functions, or sections of line bundles, defined on the manifolds.

The main theorems (cf [1]) are formulated as saying that certain bundles of vector spaces of square integrable holomorphic functions, or sections of line bundles, have positive curvature. Even though this may appear quite different from classical Brunn-Minkowski theory, it contains the Brunn-Minkowski theorem as the special case when there is enough (toric) symmetry. A prototype of the theorem, for bundles of holomorphic forms on compact manifolds, appear already in the work of Griffiths, [2], on variations of Hodge structures, but is here extended to non compact manifolds and more general line bundles.

Apart from explaining the general set up of the theory, and its relation to convex geometry, we will indicate some applications. These are motivated either by algebraic geometry (twisted relative canonical bundles associated to a proper fibration, cf [3]), Kähler geometry (variations of Kähler metrics arising as the curvature of a positive line bundle) cf [4], or complex analysis in $\mathbb{C}^n$.

References

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[3] B Berndtsson and Mihai Păun. Bergman kernels and the pseudoeffectivity of relative canonical bundles. Duke Math. J. 145 (2008).

[4] R Berman and B Berndtsson. Convexity of the K-enrgy on the space of Kähler metrics and uniqueness of extremal metrics. Arxiv: 1405.0401