The existence of designs

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A Steiner Triple System on a set X is a collection T of 3-element subsets of X such that every pair of elements of X is contained in exactly one of the triples in T. An example considered by Plücker in 1835 is the affine plane of order three, which consists of 12 triples on a set of 9 points. Plücker observed that a necessary condition for the existence of a Steiner Triple System on a set with n elements is that n be congruent to 1 or 3 mod 6. In 1846, Kirkman showed that this necessary condition is also sufficient. In 1853, Steiner posed the natural generalisation of the question: given q and r, for which n is it possible to choose a collection Q of q-element subsets of an n-element set X such that any r elements of X are contained in exactly one of the sets in Q? There are some natural necessary divisibility conditions generalising the necessary conditions for Steiner Triple Systems. The Existence Conjecture states that for all but finitely many n these divisibility conditions are also sufficient for the existence of general Steiner systems (and more generally designs). We prove the Existence Conjecture, and more generally, we show that the natural divisibility conditions are sufficient for clique decompositions of simplicial complexes that satisfy a certain pseudorandomness condition. Our proof introduces a new technique, which we call Randomised Algebraic Construction, which provides a mechanism for taking an approximate solution to a problem (produced by the probabilistic method) and converting to an exact solution (via algebraic local modifications).