Diffuse interface models in Biology and Mechanics

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In this contribution we present the idea of joining different real-world applications coming from Biology and Fluid Mechanics under the common feature of diffuse interface problems and of the need of a suitable weak solution notion for the corresponding PDE systems.

The following fields of interest are indeed particularly challenging nowadays: tumor growth models, damage in elasto-plastic materials, two-fluids flows and liquid crystal dynamics.

The main focus of this presentation regards recent modelling and analysis of tumor growth dynamics. This has indeed recently become a major issue in applied mathematics (see, e.g., [4] and references therein). In this new approach sharp interfaces are replaced by narrow transition layers arising due to adhesive forces among the cell species. Hence, a continuum thermodynamically consistent model is introduced.

However, while numerical simulations of diffuse-interface type models for tumor growth have been carried out in several papers (cf., e.g., Chap.8 in [4]), nonetheless, a rigorous mathematical analysis of the resulting systems of PDEs is still in its infancy.

In the very recent papers [2], [3], and [6], we consider a diffuse interface model for tumor growth proposed [7]. This model consists of the Cahn-Hilliard equation for the tumor cell fraction nonlinearly coupled with a reaction-diusion equation for the nutrient-rich extracellular water volume fraction. We shall first present a result on the existence of a weak solution, then we show that the weak solution is unique and continuously depends on the initial data. Furthermore, we shall give a result on the existence of a strong solution that allows to show that any weak solution regularizes in nite time. The last results will be on the existence of the global attractor in a phase space characterized by an a priori bounded energy and on some rigorous asymptotics.

However, for instance, one may include the fluid velocity either given as a datum or satisfying a generalized Darcy's (or Brinkman's) law. In this spirit we aim to report on the recent paper [5] where we consider a diffuse interface model for tumor growth recently proposed in [1]. The resulting PDE system couples four different types of equations: a Cahn-Hilliard type equation for the tumor cells (which include proliferating and dead cells), a Darcy law for the tissue velocity field, whose divergence may be different from 0 and depend on the other variables, a transport equation for the proliferating (viable) tumor cells, and a quasi-static reaction diffusion equation for the nutrient concentration. We establish existence of weak solutions for the PDE system coupled with suitable initial and boundary conditions. In particular, the proliferation function at the boundary is supposed to be nonnegative on the set where the velocity $\mathbf{u}$ satisfies $\mathbf{u} \cdot \nu > 0$, where ν is the outer normal to the boundary of the domain. We also study a singular limit as the diffuse interface coefficient tends to zero.

Finally, we will leave some space to introduce other diffuse interface models in (Fluid) Mechanics (damage, two-phase fluids, liquid crystals), which can be treated with similar techniques.

References

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