Shadows of Hodge theory in representation theory

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The Kazhdan-Lusztig conjecture is a remarkable 1979 conjecture on the characters of simple highest weight modules over a complex semi-simple Lie algebra. It was proved in 1981 by Beilinson and Bernstein [1] and Brylinski and Kashiwara [2]. The basic paradigm established by the Kazhdan-Lusztig conjecture has proven extremely useful throughout representation theory [5].Traditional proofs of the Kazhdan-Lusztig conjecture and its generalizations rely on deep geometric tools (Deligne's theory of weights or Saito's mixed Hodge modules). Recently Elias and the author gave an algebraic proof of the Kazhdan-Lusztig conjecture [4]. The idea is to establish the existence of certain "pure Hodge structures" in an algebraic manner. The proof relies on an the theory of Soergel bimodules [6] as well as some beautiful geometric ideas of de Cataldo and Migliorini [3]. Remarkably, the methods apply to more general objects than those handled by the classical theory, thus establishing Hodge structures on objects with no obvious geometric heritage.We will present a survey of the techniques and applications of "algebraic Hodge theory" in representation theory, including applications to Kazhdan-Lusztig conjectures, Janzten conjectures and positivity conjectures. Parallels and differences to the classical geometric theory will be discussed, as well as related work (toric geometry, combinatorial geometry) and other objects in representation theory where similar approaches might be fruitful (KLR algebras, ...).

References

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