Diffusion, optimal transport, and Ricci curvature

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Starting from the pioneering paper of OTTO-VILLANI [11], the link between optimal transport and Ricci curvature in smooth Riemannian geometry has been deeply studied [8, 13]. Among the various functional and analytic applications, the point of view of Optimal Transport has played a crucial role in the LOTT-STURM-VILLANI [10, 12] formulation of a "synthetic" notion of lower Ricci curvature bound, which has been extended from the realm of smooth Riemannian manifolds to the general framework of metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$, i.e. (separable, complete) metric spaces endowed with a locally finite Borel measure $\mathfrak{m}$.

Lower Ricci curvature bounds can also be captured by the celebrated BAKRY-ÉMERY [6] approach based on Markov semigroups, diffusion operators and Γ-calculus for local Dirichlet forms [7].

We will discuss a series of recent contributions [3, 4, 5, 9, 1] showing the link between the two approaches and the metric-variational theory of gradient flows [2] and diffusion equations. As a byproduct, when the Cheeger energy on $(X, \mathsf{d}, \mathfrak{m})$ is quadratic (or, equivalently, the Sobolev space $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is Hilbertian), we will show that the two approaches lead to essentially equivalent definitions and to a nice geometric framework suitable for deep analytic results.

References

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